Dual Spaces

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For the sake of abstractness and general applicability, we will consider that we are in a general field $\mathbb{F}$ which for example can be $\mathbb{R}$, $\mathbb{C}$ etc. Our space of vectors, is the vector space $\mathcal{V}$, which in general can be subspace of $\mathbb{F}^d$ for some $d \in \mathbb{N}$ for example.

1 Dual space

Definition 1.1 (Dual space)**.** The *dual space* $\mathcal{V}^*$ of the vector space $\mathcal{V}$, is the set of all linear transformations f from $\mathcal{V}$ to $\mathbb{F}$ denoted as $\mathcal{V}^* \coloneqq \mathscr{L}(\mathcal{V}, \mathbb{F})$.

Definition 1.2 (Linear functional)**.** An element $f : \mathcal{V} \to \mathbb{F}$ of $\mathcal{V}^*$ is called a *linear functional*, which takes in a vector $x \in \mathcal{V}$, and outputs an element in $\mathbb{F}$.

Note that if $f, g \in \mathcal{V}^*$, then $\alpha f + \beta g$ is also in $\mathcal{V}^*$ for $\alpha, \beta \in \mathbb{F}$ and would be defined as $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$ for all $x \in \mathcal{V}$. Let us look at a few examples :

1. If $\mathcal{V} = \mathcal{P}_n$ is the space of all univariate polynomial over the field $\mathbb{R}$ with degree $n \in \mathbb{N}$, then f defined as $f(p) = p(1) + 2p'(0)$, is a linear functional contained in $\mathcal{V}^* = \mathscr{L}(\mathcal{P}_n, \mathbb{R})$.

2. If $\mathcal{V} = \mathbb{R}^{m \times n}$ be the space of all real matrices of size $m \times n$, then f defined as $f(\mathbf{M}) = \mathrm{Tr}\left[\mathbf{M}\right]$, is a linear functional contained in $\mathcal{V}^* = \mathscr{L}(\mathbb{R}^{m \times n}, \mathbb{R})$.

3. If $\mathcal{V} = \mathcal{C}\left([-\pi, \pi]\right)$ is the space of all real valued continuous functions on $[-\pi, \pi]$, then f defined as $f(g) = \int_{-\pi}^{\pi} g(x)\cos(2x)dx$, is a linear functional contained in $\mathcal{V}^* = \mathscr{L}(\left[-\pi, \pi\right], \mathbb{R})$. This function essentially outputs the 2^{nd} Fourier coefficient of its input function.

2 Dual basis

Let $\mathcal{V}^*$ be a finite dimensional vector space, and $\beta = \{v_i\}_{i=1}^n$ be a basis of $\mathcal{V}$. Therefore, for to determine the action of any linear functional $f \in \mathcal{V}^*$, it is enough to know it action on the basis vectors $\{v_i\}_{i=1}^n$, i.e., it is enough to know what $\{f(v_i) \in \mathbb{F}\}_{i=1}^n$ are.

Definition 2.1 (Dual basis)**.** If $\mathcal{V}$ is finite dimensional and $\beta = \{v_i\}_{i=1}^n$ is a basis of $\mathcal{V}$, then $\beta^* = \{f_i\}_{i=1}^n$, the set of functionals defined by $f_i(v_j) = \delta_{ij} \; \forall \; i, j \in [n]$, is the corresponding basis of $\mathcal{V}^*$ also known as a dual basis of $\mathcal{V}$.

To verify that β^* is indeed a basis of $\mathcal{V}^*$, we just need to check the following

1. β^* is a linearly independent set of linear functionals, i.e., if $\sum\limits_{i=1}^n a_i f_i = f_0$ for some set $\{a_j \in \mathbb{F}\}_{j=1}^n$, then $a_j = 0 \; \forall \; j \in [n]$, where is f_0 is the 0 functional. To verify this, we just need to check this condition for all the basis elements of $\mathcal{V}$, $v \in \beta$, which gives us $\sum\limits_{i=1}^n a_i f_i(v_j) = 0 \; \forall \; j \in [n]$, and implies that $a_j = 0 \; \forall \; j \in [n]$ establishing the linear independence of the set β^*.

2. β^* spans $\mathcal{V}^*$, i.e., for any $f \in \mathcal{V}^*$, we have some set of elements $\{a_j \in \mathbb{F}\}_{j=1}^n$ such that $f(v) = \sum\limits_{i=1}^n a_i f_i(v) \; \forall \, v \in \mathcal{V}$.

   To check this, we evaluate f on basis vectors $v_j \in \beta$ for $j \in [n]$, which gives us $f(v_j) = \sum\limits_{i=1}^n a_i f_i(v_j) = \sum\limits_{i=1}^n a_i \delta_{ij} =$

$a_j \ \forall \ j \in [n]$, or, $a_j = f(v_j) \ \forall \ j \in [n]$. Therefore, if f is in the span of β^*, then it must be the case that the coefficients $a_j = f(v_j) \ \forall \ j \in [n]$. Now, if $g \coloneqq \sum_{i=1}^{n} f(v_i) f_i$, then to show that $g = f$, it is enough to show that they agree on the basis vectors $\{v_j\}$ of $\mathcal{V}$, that is $f(v_i) = g(v_i)$. This is simply true by the construction of $\{f_i\}_{i=1}^n$, therefore β^* spans $\mathcal{V}^*$

This establishes that $\beta^* = \{f_i\}$ is indeed a basis of $\mathcal{V}^*$, and any $f \in \mathcal{V}^*$ can be decomposed into the basis of $\mathcal{V}^*$ as

$$f = \sum_{i=1}^{n} f(v_i) f_i \tag{2.1}$$

where $\beta = \{v_i\}$ is the basis of $\mathcal{V}$. The existence of dual basis β^* and the relation with β also shows that $\mathcal{V}$ and $\mathcal{V}^*$ are isomorphic.

3 Transpose

The existence of the dual basis β^* helps us define the concept of *transpose* in linear algebra which is extensively used application.

Definition 3.1 (Transpose)**.** Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation from the vector space $\mathcal{V}$ to the vector space $\mathcal{W}$, then $T^\top : \mathcal{W}^* \to \mathcal{V}^*$ is a linear transformation from $\mathcal{W}^*$ to $\mathcal{V}^*$ that takes in a linear functional $g \in \mathcal{W}^*$, and outputs another linear functional $T^\top(g) \in \mathcal{V}^*$ defined as $T^\top(g)(x) = g\left(T(x)\right) \ \forall \ x \in \mathcal{V}$.

$$\begin{aligned} x &\xrightarrow{T} T(x) \xrightarrow{g \in \mathcal{W}^*} g(T(x)) \qquad \equiv \qquad x \xrightarrow{T^\top(g) \in \mathcal{V}^*} g(T(x)) \\ \mathcal{V} &\xrightarrow{T} \quad \mathcal{W} \xrightarrow{g \in \mathcal{W}^*} \mathbb{F} \qquad\qquad\qquad \mathcal{V} \xrightarrow{T^\top(g) \in \mathcal{V}^*} \mathbb{F} \end{aligned}$$

Theorem 3.2. *Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation from vector space $\mathcal{V}$ to vector space $\mathcal{W}$. Let $\beta = \{v_i\}_{i=1}^n$ and $\beta^* = \{f_i\}_{i=1}^n$ be the basis of $\mathcal{V}$ and $\mathcal{V}^*$ respectively, and $\gamma = \{w_j\}_{j=1}^m$ and $\gamma^* = \{g_j\}_{j=1}^m$ be the basis of $\mathcal{W}$ and $\mathcal{W}^*$ respectively. Let $A = [T]_\beta^\gamma$ be the matrix which transforms a vector in $\mathcal{V}$ to a vector in $\mathcal{W}$. Then $\left[T^\top\right]_{\gamma^*}^{\beta^*} = A^\top$.*

Proof. The matrix of $\left[T^\top\right]_{\gamma^*}^{\beta^*}$ takes in an element of $\mathcal{W}^*$ represented in its basis γ^*, and outputs an element in $\mathcal{V}^*$ represented in its basis β^*. Therefore, it is sufficient to represent every element in $\left\{T^\top(g_j)\right\}_{j=1}^m$ in the basis β^* of $\mathcal{V}^*$. Since any vector f in $\mathcal{V}^*$ can be expressed as a linear combination of the basis vectors β^* like in Equation (2.1), we have

$$\begin{aligned} T^\top(g_j) &= \sum_{i=1}^{n} T^\top(g_j)(v_i) f_i \\ &= \sum_{i=1}^{n} g_j(T(v_i)) f_i \end{aligned} \tag{3.1}$$

Therefore, the (i, j)-th element of the matrix $\left[T^\top\right]_{\gamma^*}^{\beta^*}$ is $g_j(T(v_i))$. Now, the elements of the matrix $[T]_\beta^\gamma$ are the representations of the basis vectors β of $\mathcal{V}$, in the basis vectors γ of $\mathcal{W}$. Therefore we have,

$$\begin{aligned} T(v_i) &= \sum_{k=1}^{m} A_{ki} w_k \\ \implies g_j(T(v_i)) &= g_j\left(\sum_{k=1}^{m} A_{ki} w_k\right) \\ &= \sum_{k=1}^{m} A_{ki} g_j(w_k) \qquad (\because g \text{ is a linear functional}) \\ &= A_{ji} \qquad (\because g_j(w_k) = \delta_{jk}) \end{aligned} \tag{3.2}$$

Using Equation (3.2) in Equation (3.1), we get

$$T^\top(g_j) = \sum_{i=1}^{n} A_{ji} f_i \tag{3.3}$$

which shows us that the (i, j)-th element of the matrix $\left[T^\top\right]_{\gamma^*}^{\beta^*}$ is nothing but the (j, i)-th element of A, which implies that the matrix $\left[T^\top\right]_{\gamma^*}^{\beta^*} = A^\top$. □

4 Double dual

If $\mathcal{V}$ is a vector space, then its dual space $\mathcal{V}^*$ is also a vector space. We can again define the set of all linear transformations from $\mathcal{V}^*$ to $\mathbb{F}$ as its dual space $\mathcal{V}^{**} = \mathscr{L}(\mathcal{V}^*, \mathbb{F})$, which makes it the double dual space of $\mathcal{V}$. Unlike $\mathcal{V}^*$ where we needed to define basis, there is an elegant way to go from $\mathcal{V}$ to $\mathcal{V}^{**}$ via an isomorphism Ψ. If $x \in \mathcal{V}$, and $\hat{x} = \Psi(x) \in \mathcal{V}^{**}$ acts on a linear functional $f \in \mathcal{V}^*$ and evaluates f at x returning an element in the field $\mathbb{F}$, i.e.,

$$f \xrightarrow{\hat{x} \in \mathcal{V}^{**}} f(x)$$
$$\mathcal{V}^* \xrightarrow{\hat{x} \in \mathcal{V}^{**}} \quad \mathbb{F}$$

Theorem 4.1. *If $\mathcal{V}$ is a vector space and is finite dimensional, then $\Psi : \mathcal{V} \to \mathcal{V}^{**}$ is an isomorphism.*

Proof. First, we show that Ψ is linear. If $x, y \in \mathcal{V}$, $f \in \mathcal{V}^*$, and $c \in \mathbb{F}$, then

$$\begin{aligned} \Psi(x + cy)(f) &= f(x + cy) \\ &= f(x) + cf(y) \\ &= (\Psi(x) + c\Psi(y))(f) \end{aligned}$$

Next, we show that Ψ is one to one. Suppose $\hat{x} = 0$, the zero functional, then $x = 0$. Let $\beta = \{v_i\}_{i=1}^n$, then $x = \sum\limits_{i=1}^{n} a_i v_i$ for some $\{a_i \in \mathbb{F}\}$. Therefore, if $\Psi(x) = \hat{x} = 0$, then

$$\begin{aligned} \Psi(x) &= \Psi\left(\sum_{i=1}^{n} a_i v_i\right) \\ &= \sum_{i=1}^{n} a_i \Psi(v_i) \\ \implies \Psi(x)(f) &= \sum_{i=1}^{n} a_i \Psi(v_i)(f) \qquad \forall\ f \in \mathcal{V}^* \\ &= \sum_{i=1}^{n} a_i f(v_i) \end{aligned}$$

For $f \in \{f_i\}_{i=1}^n$, we get $a_i = 0\ \forall\ i \in [n]$. This implies that $x = \sum\limits_{i=1}^{n} a_i v_i = 0$. Since $\dim(\mathcal{V}) = \dim(\mathcal{V}^{**})$, we have that Ψ is one to one and therefore an isomorphism. □

The isomorphism between $\mathcal{V}$ and $\mathcal{V}^{**}$, Ψ, allows us to map each element in $\mathcal{V}$ to a unique element in $\mathcal{V}^{**}$.

5 The infinite sequence of dual spaces

We can continue thinking about the dual of the parent space and the there always will exist and isomorphism between alternate dual spaces. Formally, Let us denote $\mathcal{V}^{(i+1)*} := \left(\mathcal{V}^{(i)*}\right)^*$ for all $i \in \mathbb{N}$ where $\mathcal{V}^{(0)*} := \mathcal{V}$ is a finite dimensional vector space. Let Ψ_i be the isomorphism between $\mathcal{V}^{(i-1)*}$ and $\mathcal{V}^{(i+1)*}$. Then, in general define

$\hat{x}^{(i+1)} := \Psi_i\left(\hat{x}^{(i-1)}\right)$ for $\hat{x}^{(i-1)} \in \mathcal{V}^{(i-1)*}$ when i is odd, and $\hat{f}^{(i+1)} := \Psi_i\left(\hat{f}^{(i-1)}\right)$ for $\hat{f}^{(i-1)} \in \mathcal{V}^{(i-1)*}$ when i is an even natural number, with $\hat{x}^{(0)} := x \in \mathcal{V}$, and $\hat{f}^{(1)} := f \in \mathcal{V}^*$. Then by the property of these isomorphisms we have

$$\hat{x}^{(i+1)}\left(\hat{f}^{(i)}\right) = \Psi_i\left(\hat{x}^{(i-1)}\right)\left(\hat{f}^{(i)}\right) = \hat{f}^{(i)}\left(\hat{x}^{(i-1)}\right) = \ldots = \hat{x}^{(2)}\left(\hat{f}^{(1)}\right) = \hat{x}^{(2)}(f)$$

when i is odd, and

$$\hat{f}^{(i+1)}\left(\hat{x}^{(i)}\right) = \Psi_i\left(\hat{f}^{(i-1)}\right)\left(\hat{x}^{(i)}\right) = \hat{x}^{(i)}\left(\hat{f}^{(i-1)}\right) = \ldots = f^{(1)}(x) = f(x)$$

when i is an even natural number. However, note that $\hat{x}^{(2)}(f)$ and $f(x)$ are the same quantities in $\mathbb{F}$, the difference lies in the order of application of these linear functionals, which makes them the dual representation of each other.